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The Side-Vertex Method for Interpolation in Triangles*

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Interpolation schemes which assume prescribed values on the boundary of a triangle are presented. The development of these interpolants is based upon univariate interpolation along line segments joining a vertex and a side. Initially, methods which only interpolate to function values on the boundary are described. This is followed by the application of several techniques which extend these methods so as to include interpolation to first order derivatives on the boundary.

1. INTRODUCTION

The purpose of this report is to present some new methods for interpolating to function values and derivatives given on the boundary of a triangle. Interpolation methods of this type have utility in such areas as finite element analysis and computer aided geometric design.

The first methods of this type were presented by Barnhill, Birkhoff and Gordon [3]. Their methods are based upon the combination of interpolation operators consisting of univariate interpolation along lines parallel to the sides of the triangle. The fundamental operators of this paper consist of univariate interpolation along lines joining a vertex and its opposing side.

In Section 2, we define the basic side-vertex method, which interpolates only to position values on the boundary and describe some improved versions of it. In Section 3, we consider interpolation to both position and slope on the boundary. Two general approaches are utilized. The first is based upon the combination of operators consisting of Hermite interpolation along lines joining a vertex and a side. Following this, a general technique for extending methods which interpolate to position only, to methods which interpolate to both position and slope is described and utilized.

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2. Interpolation to Position Data Only

The original development of the side-vertex method is based upon the Boolean sum of three operators consisting of linear interpolation along lines joining a vertex and its opposing side. For the standard triangle T_s with vertices (0, 0), (0, 1) and (1, 0); the linear interpolants have the form

$$\begin{aligned} A_1{}^s[F](p,q) &= (1-p)\,F\Big(0,\frac{q}{1-p}\Big) + pF(1,0) \\ A_2{}^s[F](p,q) &= (1-q)\,F\Big(\frac{p}{1-q},0\Big) + qF(0,1) \\ A_3{}^s[F](p,q) &= (p+q)\,F\Big(\frac{p}{p+q},\frac{q}{p+q}\Big) + (1-p-q)\,F(0,0) \end{aligned} \tag{2.1}$$

and

$$A_i{}^s \circ A_j{}^s[F] = A_1{}^s \circ A_2{}^s \circ A_3{}^s[F] = (1-p-q)\,F(0,0) + pF(1,0) + qF(0,1), \qquad i,j = 1,2,3; \quad i \neq j.$$

Therefore, the Boolean sum

$$\begin{aligned} A_1{}^s \oplus A_2{}^s \oplus A_3{}^s = {} & A_1{}^s + A_2{}^s + A_3{}^s - A_1{}^s \circ A_2{}^s \\ & - A_1{}^s \circ A_3{}^s - A_2{}^s \circ A_3{}^s + A_1{}^s \circ A_2{}^s \circ A_3{}^s \end{aligned}$$

yields the interpolation operator

$$\begin{aligned} A^s[F](p,q) = {} & (1-p)\,F\Big(0,\frac{q}{1-p}\Big) + (1-q)\,F\Big(\frac{p}{1-q},0\Big) \\ & + (p+q)\,F\Big(\frac{p}{p+q},\frac{q}{p+q}\Big) \\ & - pF(1,0) - qF(0,1) - (1-p-q)\,F(0,0). \end{aligned} \tag{2.2}$$

This operator has been discussed by Marshall [8], Marshall and Mitchell [9] and Barnhill [1].

For an arbitrary triangle T with vertices $V_i = (x_i, y_i)$, $i = 1, 2, 3$ an analogous operator can be defined by the use of an affine map

$$\begin{aligned} x &= x(p,q) \\ y &= y(p,q) \end{aligned}$$

which maps T_s to T. For F defined on T, we define

$$A[F](x,y) = A^s[\hat{F}](p(x,y), q(x,y)) \tag{2.3}$$

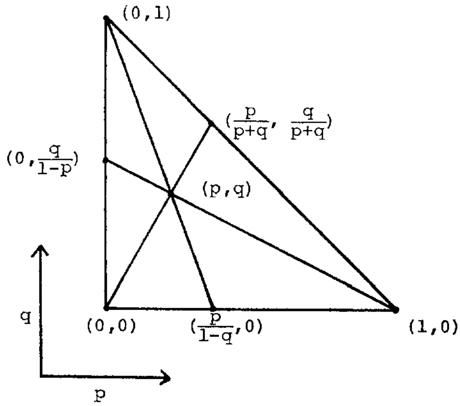

FIGURE 2.1

where

$$\hat{F}(p, q) = F(x(p, q), y(p, q))$$

and $p = p(x, y)$, $q = q(x, y)$ represents the inverse of the affine map.

There are, of course, a total of six transformations depending upon the association of the vertices of T_s and T. In general, it is possible that an interpolation scheme defined on the standard triangle could lead to six different schemes for an arbitrary triangle. These schemes are said to be *affine equivalent* to each other and any method for which all the affine equivalents are identical is termed an *affine invariant* method. Each of these affine transformations may be represented by

$$\begin{aligned} x(p, q) &= x_i p + x_j q + x_k r \\ y(p, q) &= y_i p + y_j q + y_k r \end{aligned} \tag{2.4}$$

where $r = 1 - p - q$ and (i, j, k) is one of the six permutations of $(1, 2, 3)$. The associated inverse mapping is given by

$$\begin{aligned} p(x, y) &= \frac{\begin{vmatrix} x - x_k & x_j - x_k \\ y - y_k & y_j - y_k \end{vmatrix}}{\begin{vmatrix} x_i - x_k & x_j - x_k \\ y_i - y_k & y_j - y_k \end{vmatrix}} \\ q(x, y) &= \frac{\begin{vmatrix} x - x_i & x_i - x_k \\ y - y_i & y_i - y_k \end{vmatrix}}{\begin{vmatrix} x_j - x_k & x_i - x_k \\ y_j - y_k & y_i - y_k \end{vmatrix}} \\ r(x, y) &= \frac{\begin{vmatrix} x - x_j & x_i - x_j \\ y - y_j & y_i - y_j \end{vmatrix}}{\begin{vmatrix} x_k - x_j & x_i - x_j \\ y_k - y_j & y_i - y_j \end{vmatrix}}. \end{aligned} \tag{2.5}$$

Applying equation (2.3) for each transformation, we obtain the following interpolant for F defined on T

$$\begin{aligned}A_{ijk}[F](x, y) = {} & (1 - p(x, y))\, F\left(\frac{x - x_i p(x, y)}{1 - p(x, y)}, \frac{y - y_i p(x, y)}{1 - p(x, y)}\right) \\ & + (1 - q(x, y))\, F\left(\frac{x - x_j q(x, y)}{1 - q(x, y)}, \frac{y - y_j q(x, y)}{1 - q(x, y)}\right) \\ & + (1 - r(x, y))\, F\left(\frac{x - x_k r(x, y)}{1 - r(x, y)}, \frac{y - y_k r(x, y)}{1 - r(x, y)}\right) \\ & - p(x, y)\, F(x_i, y_i) - q(x, y)\, F(x_j, y_j) - r(x, y)\, F(x_k, y_k).\end{aligned} \tag{2.6}$$

From equation (2.4) and (2.5), it is clear that each permutation leads to the same formula and consequently this method is affine invariant. We will find it convenient to utilize the barycentric coordinates b_i, $i = 1, 2, 3$ defined by

$$\begin{aligned} x &= b_1 x_1 + b_2 x_2 + b_3 x_3 \\ y &= b_1 y_1 + b_2 y_2 + b_3 y_3 \\ 1 &= b_1 + b_2 + b_3 \end{aligned} \tag{2.7}$$

and

$$b_i = b_i(x, y) = \frac{\begin{vmatrix} x - x_j & x - x_k \\ y - y_j & y - y_k \end{vmatrix}}{\begin{vmatrix} x_i - x_j & x_i - x_k \\ y_i - y_j & y_i - y_k \end{vmatrix}}, \quad i = 1, 2, 3; \quad i \neq j \neq k \neq i.$$

We will denote the point opposite the vertex V_i by

$$S_i = S_i(x, y) = \left(\frac{x - x_i b_i}{1 - b_i}, \frac{y - y_i b_i}{1 - b_i}\right), \qquad i = 1, 2, 3. \tag{2.8}$$

This notation is further illustrated in Figure 2.2.

Incorporating this notation, the basic side-vertex method takes the rather simple form

$$A[F] = \sum_{i=1}^{3} [(1 - b_i)\, F(S_i) - b_i F(V_i)]. \tag{2.9}$$

Using the properties

$$\begin{aligned} b_j(S_i) &= \begin{cases} \dfrac{b_j}{1 - b_i} & i \neq j \\ 0 & i = j \end{cases} \\ b_j(V_i) &= \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases} \end{aligned} \tag{2.10}$$

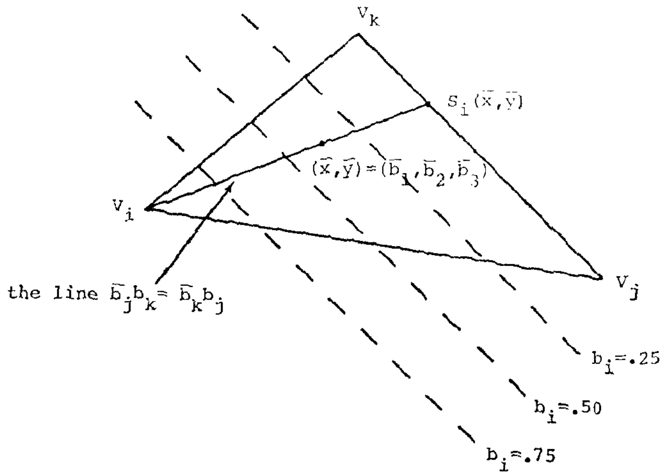

FIGURE 2.2

we can easily compute

$$\begin{aligned} A[b_j] &= \sum_{i=1}^{3} [(1 - b_i)\, b_j(S_i) - b_i b_j(V_i)] \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{3} \left[(1 - b_i) \frac{b_j}{(1 - b_i)}\right] - b_j \\ &= b_j\,. \end{aligned}$$

Since the linear span$\langle 1, x, y\rangle$ is equivalent to $\langle b_1, b_2, b_3\rangle$, we can conclude that A has algebraic precision of degree at least one. A has no higher degree of precision, because

$$A[b_j b_k] = \frac{b_j b_k}{b_j + b_k} \qquad j \neq k$$

and

$$A[b_i^2] = \frac{b_i^3 - b_i b_j b_k}{(1 - b_j)(1 - b_k)} \qquad i \neq j \neq k \neq i. \tag{2.11}$$

Concerning the continuity of $A[F]$ we can note that the only potential discontinuity is in the term $(1 - b_i)\, F(S_i)$ at $(x, y) = V_i$. Since

$$|(1 - b_i)\, F(S_i)| \leqslant |(1 - b_i)|\, \| F \|_T$$

it follows that

$$\lim_{b_i \to 1} (1 - b_i)\, F(S_i) = 0$$

and so $A[F] \in C(T)$ for $F \in C(T)$. But this is the extent of the smoothness of $A[F]$. In general, $A[F]$ will have singularities in its first (and higher) order derivatives at the vertices, regardless of the smoothness of F. Equation (2.11) provides an example of this. If we compute

$$\frac{\partial}{\partial x}\left[\frac{b_j b_k}{b_j + b_k}\right] = \frac{b_k^2 \dfrac{\partial b_j}{\partial x} + b_j^2 \dfrac{\partial b_k}{\partial x}}{(b_j + b_k)^2}$$

and let (x, y) approach V_i along the line $\alpha b_k = \beta b_j$ we find that

$$\frac{\partial}{\partial x}\left[\frac{b_j b_k}{b_j + b_k}\right]\Bigg|_{V_i} = \frac{\beta^2 \dfrac{\partial b_j}{\partial x} + \alpha^2 \dfrac{\partial b_k}{\partial x}}{(\alpha + \beta)^2},$$

which is dependent upon the direction of approach.

We now proceed to improve upon the operator A. It is interesting to note that if α is any continuous function defined on $[0, 1]$ with the property $\alpha(0) = 0$, $\alpha(1) = 1$, then the following generalization of A will be an interpolation operator for $F \in C(T)$:

$$\tilde{A}[F] = \sum_{i=1}^{3} [\alpha(1 - b_i)\, F(S_i) - \alpha(b_i)\, F(V_i)]. \tag{2.12}$$

Of particular interest is the case $\alpha(t) = t^2$ which leads to

$$A^*[F] = \sum_{i=1}^{3} [(1 - b_i)^2\, F(S_i) - b_i^2 F(V_i)] \tag{2.13}$$

which defines an operator with algebraic precision of degree two with the property that $A^*[F] \in C^1(T)$ for $F \in C^1(T)$. This operator has been developed by Thomas [10] as the Boolean sum of the operators

$$A_i^*[F] = (1 - b_i)^2\, F(S_i), \qquad i = 1, 2, 3.$$

In order to verify that $A^*[F] \in C^1(T)$ for $F \in C^1(T)$ we need only concern ourselves with the term $(1 - b_i)^2\, F(S_i)$ at the vertex V_i. Performing the differentiation, we find that

$$\begin{aligned}\frac{\partial[(1 - b_i)^2\, F(S_i)]}{\partial x} &= (1 - b_i)\, F_x(S_i) + [(x - x_i)\, F_x(S_i) \\ &\quad + (y - y_i)\, F_y(S_i) - 2(1 - b_i)\, F(S_i)]\, \frac{\partial b_i}{\partial x}.\end{aligned}$$

Using the fact that $F, F_x, F_y \in C(T)$, it is easy to see that $\partial[(1 - b_i)^2\, F(S_i)]/\partial x$ is continuous and that

$$\lim_{b_i \to 1} \frac{\partial[(1 - b_i)^2\, F(S_i)]}{\partial x} = 0.$$

In general, the second derivatives of $A^*[F]$ exhibit a similar behavior to the first derivatives of $A[F]$ in that the values at the vertices depend upon the direction of approach.

Concerning the algebraic precision of A^*, we first note that $\langle 1, x, y, x^2, xy, y^2\rangle = \langle b_1, b_2, b_3, b_1^2, b_2^2, b_3^3\rangle = \langle b_1, b_2, b_3, b_1b_2, b_1b_3, b_2b_3\rangle$. Using equation (2.10) we find that

$$\begin{aligned} A^*[b_j] &= \sum_{i=1}^{3} (1-b_i)^2\, b_j(S_i) - b_i^2 b_j(V_i) \\ &= \sum_{\substack{i=1\\ i\neq j}}^{3} \left[(1-b_i)^2 \frac{b_j}{(1-b_i)}\right] - b_j^2 \\ &= b_j(1+b_j) - b_j^2 \\ &= b_j \end{aligned}$$

and

$$\begin{aligned} A^*[b_jb_k] &= \sum_{i=1}^{3} (1-b_i)^2 b_j(S_i)\, b_k(S_i) - b_i^2 b_j(V_i)\, b_k(V_i) \\ &= (1-b_i)^2 \frac{b_j}{(1-b_i)} \frac{b_k}{(1-b_i)} = b_jb_k, \qquad i \neq j, k; \end{aligned}$$

and so A^* has precision of degree at least two.

The operator A^* has no higher algebraic precision since

$$A^*[b_i^3] = \frac{b_i^4 - b_i^2 b_j b_k}{(1-b_j)(1-b_k)}, \qquad i \neq j \neq k \neq i$$

and

$$A^*[b_i^2 b_j] = \frac{b_i^2 b_j}{b_i + b_j}, \qquad i \neq j \neq k \neq i.$$

In fact, it is true in general that an interpolation operator which utilizes only values of F on ∂T (and not derivatives) cannot be exact for all third degree polynomials. This is due to the fact that $b_1b_2b_3$ is a cubic which is identically zero on ∂T.

3. Interpolation to Both Position and Slope

In this section, we extend the methods of the previous section to include interpolation to first order derivatives on the boundary of T. Two general approaches are used and several methods are obtained. The first approach is based upon operators similar to A_i, $i = 1, 2, 3$; in that they consist of univariate interpolation along line segments joining a vertex and its opposing edge. Rather than linear interpolation, we use the general Hermite operator

$$H[g](t) = h(t)\,g(1) + \bar{h}(t)\,g'(1) + h(1-t)\,g(0) - \bar{h}(1-t)\,g'(0), \tag{3.1}$$

where $h(1) = \bar{h}'(1) = 1$, $h'(1) = h(0) = h'(0) = \bar{h}(1) = \bar{h}(0) = \bar{h}'(0) = 0$. Applying this operator to

$$R_i(t) = F(tS_i + (1-t)\,V_i), \qquad i = 1, 2, 3 \tag{3.2}$$

and letting $t = t(x, y) = 1 - b_i$, we define

$$D_i[F] = h(1-b_i)\,F(S_i) + \bar{h}(1-b_i)\,R_i'(1) + h(b_i)\,F(V_i) - \bar{h}(b_i)\,R_i'(0), \tag{3.3}$$

where

$$\begin{aligned} R_i'(1) &= \frac{(x - x_i)\,F_x(S_i) + (y - y_i)\,F_y(S_i)}{1 - b_i}, \\ R_i'(0) &= \frac{(x - x_i)\,F_x(V_i) + (y - y_i)\,F_y(V_i)}{1 - b_i}. \end{aligned} \tag{3.4}$$

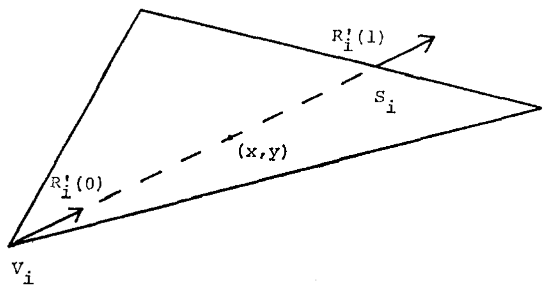

FIGURE 3.1

While we are primarily interested in the case of cubic Hermite interpolation, where $h(t) = t^2(3 - 2t)$ and $\bar{h}(t) = t^2(t - 1)$, many of our results hold in the more general situation without much additional complication. However, we do require that $h, \bar{h} \in C^2[0, 1]$. In order to simplify some later equations, we introduce the notation

$$\hat{h}(t) = \frac{\bar{h}(t)}{t(t-1)},$$

$$\frac{\partial F}{\partial e_j}(V_i) = (x_k - x_i)\,F_x(V_i) + (y_k - y_i)\,F_y(V_i), \tag{3.5}$$

$$\frac{\partial F}{\partial e_j}(S_i) = (x_k - x_i)\,F_x(S_i) + (y_k - y_i)\,F_y(S_i)$$

and write

$$D_i = B_i + P_i \tag{3.6}$$

where

$$B_i[F] = h(1-b_i)\,F(S_i) - b_i\bar{h}(1-b_i)\left[b_j \frac{\partial F}{\partial e_k}(S_i) + b_k \frac{\partial F}{\partial e_j}(S_i)\right],$$

$$P_i[F] = h(b_i)\,F(V_i) + b_i\bar{h}(b_i)\left[b_j \frac{\partial F}{\partial e_k}(V_i) + b_k \frac{\partial F}{\partial e_j}(V_i)\right], \tag{3.7}$$

$$i = 1, 2, 3; \qquad i \neq j \neq k \neq i.$$

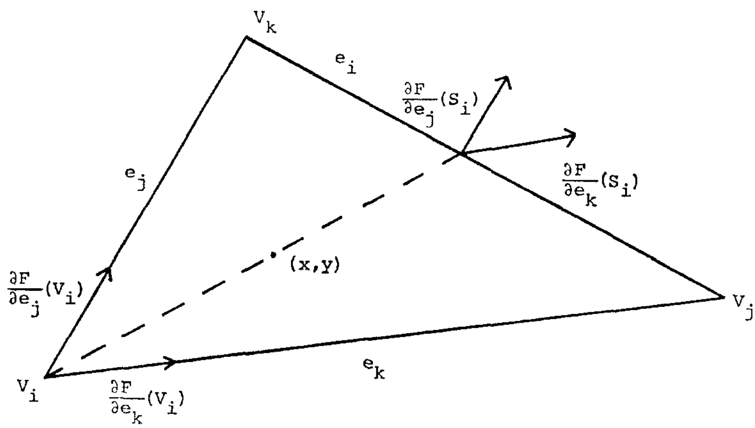

FIGURE 3.2

We are ultimately interested in approximations which are contained in $C^1(T)$. Since the first order derivatives of $D_i[F]$ on T involve second order derivatives of F on e_i, we require these derivatives to be continuous. Consequently, we define

$$C_T^{\,2} = \left\{F\colon F \in C^1(T);\ \frac{\partial^{n+m}F}{\partial x^n\,\partial y^m}\bigg|_{e_i} \in C(e_i),\ n+m=2;\ i=1,2,3\right\}.$$

Note that $C_T^{\,2}$ admits functions which are not C^2-compatible at the vertices but requires C^1-compatibility. The only potential C^1-discontinuity of $D_i[F]$ is at the vertex V_i, but since $D_i[F]$ interpolates to first order derivatives of V_i, it is clear that $D_i[F] \in C^1(T)$ for $F \in C_T^{\,2}$. In general, this is the extent of the smoothness of $D_i[F]$ regardless of the smoothness of F, h and $\bar{h}$. As an example, when $D_i = D_{ci}$ is based upon cubic Hermite interpolation, we have

$$D_{ci}[b_\ell^{\,2}b_n b_m] = \begin{cases} 0 & i=\ell,\quad \ell \neq n \neq m \neq \ell \\ \dfrac{b_\ell^{\,2}b_n b_m}{1-b_i} & i \neq \ell,\quad \ell \neq n \neq m \neq \ell \end{cases} \tag{3.8}$$

and these functions have second order derivatives whose value at V_i is dependent upon the direction of approach.

Our first method is based upon a technique due to Brown and Little [1].

THEOREM 3.1. *Let $F \in C_T^2$, then*

$$D[F] = \frac{b_2^2 b_3^2 D_1[F] + b_1^2 b_3^2 D_2[F] + b_1^2 b_2^2 D_3[F]}{b_2^2 b_3^2 + b_1^2 b_3^2 + b_1^2 b_2^2} \tag{3.9}$$

is contained in $C^1(T)$ and interpolates to F and its first order derivatives on ∂T. Also, $D[p] = p$ for all functions p such that $D_i[p] = p$, $i = 1, 2, 3$.

Proof. The weight functions

$$W_i = \frac{b_j^2 b_k^2}{b_2^2 b_3^2 + b_1^2 b_3^2 + b_1^2 b_2^2}, \qquad i = 1, 2, 3; \qquad i \neq j \neq k \neq i,$$

have the properties:

$$\sum_{i=1}^{3} W_i = 1,$$

$$W_i \,|_{e_j} = \delta_{ij},$$

$$\partial W_i \,|_{e_i} = 0;$$

where ∂ represents any first order differentiation.

This operator, for the case of cubic Hermite interpolation, was first considered by Barnhill, Herron and Little [2]. In this case, D is exact for all polynomials in the ten dimensional space of cubic polynomials

$$C = \langle b_i, b_i b_j, b_i b_j b_k ; i, j, k = 1, 2, 3 \rangle. \tag{3.10}$$

In order to develop an interpolant based upon the Boolean sum of these operators, we require the composition $D_i \circ D_j$. By inspection, we can see that $D_i \circ D_j[F]$ will involve the limit along e_j of second order derivatives of F. For example, in the case of cubic interpolation,

$$D_{ci} \circ D_{cj}[F] = \sum_{n=1}^{3} P_{cn}[F] + b_1 b_2 b_3 \times \left[6F(V_k) + 2\frac{\partial F}{\partial e_i}(V_k) + 2\frac{\partial F}{\partial e_j}(V_k) + \frac{\partial^2 F}{\partial e_i\, \partial e_j}(V_k \circ e_j)\right], \qquad i \neq j \neq k \neq i; \tag{3.11}$$

where the argument $V_k \circ e_j$ is used to indicate the value obtained at V_k as a limit along the edge e_j. For these operators to commute, so that the Boolean sum will interpolate on both e_i and e_j, it would be necessary to assume C^2-compatible data at the vertices. Rather than make this assumption, we consider the more general operators

$$\bar{D}_k = D_i + D_j - \frac{b_i D_j \circ D_i + b_j D_i \circ D_j}{b_i + b_j}, \quad k = 1, 2, 3; \quad i \neq j \neq k \neq i; \tag{3.12}$$

which reduce to the normal Boolean sum in the case of compatible data. It is easy to verify that $\bar{D}_k$ interpolates to F and its first order derivatives on e_i and e_j, and that $\bar{D}_k$ is Hermite interpolation on e_k. Using a technique originally due to Gregory [7], we obtain our next interpolant.

THEOREM 3.2. *If $F \in C_T^2$, then*

$$\bar{D}[F] = W_1 \bar{D}_1[F] + W_2 \bar{D}_2[F] + W_3 \bar{D}_3[F] \tag{3.13}$$

where

$$W_i = b_i^2(3 - 2b_i + 6b_j b_k), \qquad i \neq j \neq k \neq i, \tag{3.14}$$

is contained in $C^1[T]$ and interpolates to F and its first order derivatives on ∂T. Furthermore, $\bar{D}[p] = p$ for all functions p such that $D_i[p] = p$, $i = 1, 2, 3$.

Proof. As Gregory has shown, the weight functions have the properties:

$$\sum_{i=1}^{3} W_i = 1,$$

$$W_i |_{e_i} = 0,$$

$$\partial W_i |_{e_i} = 0,$$

where ∂ represents any first order differentiation.

While the weight functions given by equation (3.14) are the simplest polynomials with the required properties, there are many other functions which will also satisfy these conditions. For example

$$W_i = \frac{b_i^2}{b_1^2 + b_2^2 + b_3^2},$$

$$W_i = b_i^2 \left[\frac{3b_k + b_i}{b_i + b_k} + \frac{3b_j + b_i}{b_j + b_k} - 1 \right], \qquad i \neq j \neq k \neq i$$

or even

$$W_i = \tfrac{1}{2}\{(1 - b_i)[1 - \cos(\pi b_i)] + b_i[\cos(\pi b_j) + \cos(\pi b_k)]\}, \quad i \neq j \neq k \neq i.$$

THEOREM 3.3. *Let $F \in C_T^2$ and define*

$$D' = D_1 - \frac{b_2 b_3}{\Delta} D_1 \circ (D_2 + D_3) + D_2 - \frac{b_1 b_3}{\Delta} D_2 \circ (D_1 + D_3)$$
$$+ D_3 - \frac{b_1 b_2}{\Delta} D_3 \circ (D_1 + D_2) \tag{3.15}$$

where $\Delta = b_1 b_2 + b_1 b_3 + b_2 b_3$. Then $D'[F] \in C^1(T)$ and interpolates to F, F_x and F_y on ∂T. The precision consists of all functions p such that $D_i[p] = p$, $i = 1, 2, 3$.

Proof. On e_i, where $b_i = 0$,

$$D'[F]|_{e_i} = F|_{e_i} + D_j[F]|_{e_i} + D_k[F]|_{e_i} - D_i \circ (D_j + D_k)[F]|_{e_i} = F|_{e_i}, \quad i \neq j \neq k \neq i.$$

Let ∂ represent any first order differentiation, then

$$\partial D' = \partial D_1 - \frac{b_2 b_3}{\Delta} \partial[D_1 \circ (D_2 + D_3)] - D_1 \circ (D_2 + D_3)\, \partial \left[\frac{b_2 b_3}{\Delta}\right]$$
$$+ \partial D_2 - \frac{b_1 b_3}{\Delta} \partial[D_2 \circ (D_1 + D_3)] - D_2 \circ (D_1 + D_3)\, \partial \left[\frac{b_1 b_3}{\Delta}\right]$$
$$+ \partial D_3 - \frac{b_1 b_2}{\Delta} \partial[D_3 \circ (D_1 + D_2)] - D_3 \circ (D_1 + D_2)\, \partial \left[\frac{b_1 b_2}{\Delta}\right].$$

Again on the edge e_i, and using the fact that $D_i \circ D_j[F]$ is Hermite interpolation, H, on any edge, we obtain

$$\partial D'[F]|_{e_i} = \partial F|_{e_i} + \partial D_j[F]|_{e_i} + \partial D_k[F]|_{e_i} - \partial[D_i \circ (D_j + D_k)][F]|_{e_i}$$
$$- H\, \partial \left[\frac{b_1 b_2 + b_1 b_3 + b_2 b_3}{\Delta}\right]\Bigg|_{e_i} = \partial F|_{e_i},$$

The precision claimed follows immediately by direct substitution.

As an application of this theorem, we choose cubic Hermite interpolation and use equation (3.11) to obtain

$$D_c'[F] = \sum_{i=1}^{3} [B_{ci}[F] - P_{ci}[F] - C_i'[F]] \tag{3.16}$$

where

$$C_i'[F] = \frac{(1 - b_i)\, b_i^2 b_j b_k}{\Delta} \left[6F(V_i) + 2 \frac{\partial F}{\partial e_j}(V_i) + 2 \frac{\partial F}{\partial e_k}(V_i) \right.$$
$$\left. + \frac{b_j}{b_j + b_k} \frac{\partial^2 F}{\partial e_j\, \partial e_k}(V_i \circ e_j) + \frac{b_k}{b_j + b_k} \frac{\partial^2 F}{\partial e_j\, \partial e_k}(V_i \circ e_k)\right],$$
$$i \neq j \neq k \neq i. \tag{3.17}$$

We now return to more discussion on Boolean sums. While it is not the case that $\bar{D}_i$ and $\bar{D}_j$ commute, we do know that $\bar{D}_i \oplus D_i$, $i = 1, 2, 3$; will have the interpolatory properties of $\bar{D}_i$ in that it will interpolate to F and its first order derivatives on e_j and e_k. Since $\bar{D}_i[F]$ and any $D_n \circ D_m[F]$, $n \neq m$ reduce to Hermite interpolation on e_i, we also know that $\bar{D}_i \oplus D_i$ will interpolate to F (but not necessarily its first order derivatives) on e_i. This leads to the following theorem.

THEOREM 3.4. *Let $F \in C_T{}^2$ and define*

$$D^* = \sum_{i=1}^{3} b_i[\bar{D}_i \oplus D_i], \tag{3.18}$$

Then $D^*[F] \in C^1(T)$ and interpolates to F, F_x and F_y on ∂T. The precision set is the intersection of precision sets of D_i, $i = 1, 2, 3$.

Proof. Because of the interpolatory properties of $\bar{D}_i \oplus D_i$ stated above, it is clear that $D^*[F]$ interpolates to F on ∂T. Since

$$\begin{aligned} \partial D^*[F] = {} & b_1\, \partial[\bar{D}_1 \oplus D_1] + [\bar{D}_1 \oplus D_1]\, \partial b_1 \\ & + b_2\, \partial[\bar{D}_2 \oplus D_2] + [\bar{D}_2 \oplus D_2]\, \partial b_2 \\ & + b_3\, \partial[\bar{D}_3 \oplus D_3] + [\bar{D}_3 \oplus D_3]\, \partial b_3 \end{aligned}$$

where ∂ represents any first order differentiation, it is also clear that the first order derivatives of $D^*[F]$ coincide with those of F on ∂T. Again precision is just a matter of direct substitution.

As an example, we choose $D_i = D_{ci}$, $i = 1, 2, 3$. We expand equation (3.18) to obtain

$$\begin{aligned} D_c^* = {} & D_{c1} - \frac{b_2 b_3(1 + b_1)}{b_1 + b_2 b_3} D_{c1} \circ (D_{c2} + D_{c3}) \\ & + D_{c2} - \frac{b_1 b_3(1 + b_2)}{b_2 + b_1 b_3} D_{c2} \circ (D_{c1} + D_{c3}) \\ & + D_{c3} - \frac{b_1 b_2(1 + b_3)}{b_3 + b_1 b_2} D_{c3} \circ (D_{c1} + D_{c2}) \end{aligned} \tag{3.19}$$

and use equation (3.11) to obtain

$$D_c^*[F] = \sum_{i=1}^{3} [B_{ci}[F] - P_{ci}[F] - C_i^*[F]] \tag{3.20}$$

where

$$C_i^*[F] = \frac{b_i^2 b_j b_k(1 - b_j b_k)}{b_i + b_j b_k}\left[6F(V_i) + 2\frac{\partial F}{\partial e_j}(V_i) + 2\frac{\partial F}{\partial e_k}(V_i)\right]$$

$$+ \frac{b_i^2 b_j^2 b_k(1 + b_k)}{b_k + b_i b_j}\frac{\partial^2 F}{\partial e_j\, \partial e_k}(V_i \circ e_j)$$

$$+ \frac{b_i^2 b_k^2 b_j(1 + b_j)}{b_j + b_i b_k}\frac{\partial^2 F}{\partial e_j\, \partial e_k}(V_i \circ e_k), \qquad i \neq j \neq k \neq i. \quad (3.21)$$

It is interesting to note the similarity of equation (3.19) and equation (3.15).

THEOREM 3.5. *Let $\Phi[F] \in C_T^2$ satisfy the fifteen conditions:*

$$\Phi[F](V_i) = F(V_i), \qquad i = 1, 2, 3;$$

$$\frac{\partial \Phi[F]}{\partial e_i}(V_j) = \frac{\partial F}{\partial e_i}(V_j), \qquad i = 1, 2, 3, \quad i \neq j; \quad (3.22)$$

$$\frac{\partial^2 \Phi[F]}{\partial e_i\, \partial e_j}(V_k \circ e_j) = \frac{\partial^2 F}{\partial e_i\, \partial e_j}(V_k \circ e_j), \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k \neq i.$$

Then

$$\hat{D}[F] = \sum_{i=1}^{3} [B_i[F] - B_i \circ \Phi[f]] + \Phi[F] \quad (3.23)$$

is contained in $C^1(T)$ and interpolates to F, F_x and F_y on ∂T. Furthermore, $\hat{D}$ has the precision of Φ.

Proof. Consider the subspace

$$C_T^2(\Phi) = \{\hat{F} : \hat{F} = F - \Phi[F], F \in C_T^2\}.$$

On this subspace, $D_i[\hat{F}] = B_i[\hat{F}]$, $i = 1, 2, 3$. Since $D_i \circ D_j[\hat{F}]$, $i \neq j$ only involves the values of $\hat{F}$ and its derivatives at the vertices, $D_i \circ D_j[\hat{F}] = D_j \circ D_i[\hat{F}] = 0$. Therefore, the triple Boolean sum over $C_T^2(\Phi)$ reduces to

$$D_1 \oplus D_2 \oplus D_3[\hat{F}] = B_1[\hat{F}] + B_2[\hat{F}] + B_3[\hat{F}].$$

Consequently,

$$\sum_{i=1}^{3} B_i[F - \Phi[F]]$$

will interpolate to $F - \Phi[F]$ on ∂T. This implies that $\hat{D}[F]$ will interpolate to F. The statement about precision is obvious.

As a source of interpolants with the properties of Φ, we can use the following.

LEMMA 3.6. *The interpolant*

$$\hat{\Phi}[F] = b_1\left[\frac{b_2D_2 \circ D_3[F] + b_3D_3 \circ D_2[F]}{b_2 + b_3}\right] + b_2\left[\frac{b_1D_1 \circ D_3[F] + b_3D_3 \circ D_1[F]}{b_1 + b_3}\right]$$
$$+ b_3\left[\frac{b_1D_1 \circ D_2[F] + b_2D_2 \circ D_1[F]}{b_1 + b_2}\right] \tag{3.24}$$

satisfies the fifteen conditions of equation (3.22).

Proof. Since $D_i \circ D_j[F]$, $i \neq j$ is Hermite interpolation on ∂T, it is clear that the first nine conditions are satisfied. For the remaining six conditions, we let $P_{ij} = D_i \circ D_j - P_1 - P_2 - P_3$, $i \neq j$, and wirte

$$\hat{\Phi}[F] = P_1[F] + P_2[F] + P_3[F] + \sum_{\substack{i=1\\ i\neq j\neq k\neq i}}^{3} b_i\left[\frac{b_jP_{jk}[F] + b_kP_{kj}[F]}{b_j + b_k}\right]. \tag{3.25}$$

Let us consider one of the conditions of equations (3.22), say $i = 1, j = 2$, $k = 3$. Applying this to one of the last six terms of equation (3.25), we find that

$$\frac{\partial^2\left[\dfrac{b_ib_jP_{jk}[F]}{b_j + b_k}\right]}{\partial e_1\,\partial e_2}(V_3 \circ e_2) = P_{jk}[F](V_3)\frac{\partial^2\left[\dfrac{b_ib_j}{b_j + b_k}\right]}{\partial e_1\,\partial e_2}(V_3 \circ e_2)$$
$$+ \frac{\partial\left[\dfrac{b_ib_j}{b_i + b_k}\right]}{\partial e_1}(V_3)\frac{\partial P_{jk}[F]}{\partial e_2}(V_3)$$
$$+ \frac{\partial P_{jk}[F]}{\partial e_1}(V_3)\frac{\partial\left[\dfrac{b_ib_j}{b_j + b_k}\right]}{\partial e_2}(V_3)$$
$$+ \left[\frac{b_ib_j}{b_j + b_k}\right]\bigg|_{V_3\circ e_2}\frac{\partial^2P_{jk}[F]}{\partial e_1\,\partial e_2}(V_3 \circ e_2).$$

The fact that $P_{jk}[F] = 0$ on ∂T implies that

$$\frac{\partial^2\left[\dfrac{b_ib_jP_{jk}[F]}{b_j + b_k}\right]}{\partial e_1\,\partial e_2}(V_3 \circ e_2) = \left[\frac{b_ib_j}{b_j + b_k}\right]\bigg|_{V_3\circ e_2}\left[\frac{\partial^2P_{jk}[F]}{\partial e_1\,\partial e_2}(V_3 \circ e_2)\right].$$

Let $i \neq j \neq k \neq i$ and $\ell \neq n$, then

$$\left[\frac{b_i b_j}{b_j + b_k}\right]\Bigg|_{V_\ell \circ e_n} = \begin{cases} 1 & i = \ell, \quad k = n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{\partial^2 \hat{\Phi}[F]}{\partial e_1 \, \partial e_2}(V_3 \circ e_2) = \frac{\partial^2 P[F]}{\partial e_1 \, \partial e_2}(V_3 \circ e_2) + \frac{\partial^2 [D_1 \circ D_2[F] - P]}{\partial e_1 \, \partial e_2}(V_3 \circ e_2)$$

where $P = P_1 + P_2 + P_3$. Since it can be shown that

$$\frac{\partial^2 D_1 \circ D_2[F]}{\partial e_1 \, \partial e_2}(V_3 \circ e_2) = \frac{\partial^2 F}{\partial e_1 \, \partial e_2}(V_3 \circ e_2)$$

the argument is complete.

As an example of the use of this lemma, we apply it to the case of cubic Hermite interpolation and obtain

$$\begin{aligned} \hat{\Phi}_c[F] = \sum_{\substack{i=1 \\ i \neq j \neq k \neq i}}^{3} b_i^2 \Big[& (3 - 2b_i + 6b_j b_k) F(V_i) \\ & + b_k(1 + 2b_j) \frac{\partial F}{\partial e_j}(V_i) + b_j(1 + 2b_k) \frac{\partial F}{\partial e_k}(V_i) \\ & + \frac{b_j b_k^2}{b_j + b_k} \frac{\partial^2 F}{\partial e_j \, \partial e_k}(V_i \circ e_j) + \frac{b_k b_j^2}{b_k + b_j} \frac{\partial^2 F}{\partial e_j \, \partial e_k}(V_i \circ e_k) \Big]. \end{aligned} \tag{3.26}$$

It is interesting to note that this interpolant is the unique element of the Birkhoff and Mansfield [6] space Φ_{15} determined by the conditions of equation (3.22). In the case of C^2-compatibility, this interpolant is Birkhoff's tricubic interpolation [4].

We now illustrate the use of Theorem 3.5. We choose $D_i = D_{qi} = B_{qi} + P_{qi}$ to be based upon

$$h(t) = t^3(4 - 3t), \qquad \bar{h}(t) = t^3(t - 1),$$

and $\Phi = \hat{\Phi}_c$ of equation (3.26). As it turns out,

$$\sum_{i=1}^{3} B_{qi}[\hat{\Phi}_c] = \hat{\Phi}_c + \sum_{i=1}^{3} [P_{qi}[F] + Q_i[F]] \tag{3.27}$$

where

$$\begin{aligned} Q_i[F] = b_i^2 b_j b_k \Big[& 6F(V_i) + \frac{\partial F}{\partial e_j}(V_i) + \frac{\partial F}{\partial e_k}(V_i) \\ & + \frac{b_j}{b_j + b_k} \frac{\partial^2 F}{\partial e_j \, \partial e_k}(V_i \circ e_j) + \frac{b_k}{b_j + b_k} \frac{\partial^2 F}{\partial e_j \, \partial e_k}(V_i \circ e_k) \Big], \\ & i \neq j \neq k \neq i. \end{aligned} \tag{3.28}$$

Therefore, Theorem 3.5 yields the following interpolation operator

$$\hat{D}_q = \sum_{i=1}^{3} [B_{qi} - P_{qi} - Q_i]. \tag{3.29}$$

This approximation has the interesting property that in the case of C^2-compatibility, all of the weight functions are polynomials.

Our second approach is based on the following technique: We assume that G is any interpolation operator that utilizes only position information and w is a function which is zero on all of ∂T. We then consider interpolants of the form

$$P[F](x, y) = G[F](x, y) + w(x, y)\, h(x, y)$$

where h is yet to be determined. Imposing conditions that require $P[F]$ to interpolate to first order derivatives on ∂T will impose only interpolation to position data on h. For example, say $T = T_s$ and we impose interpolation to the normal derivative along the edge $p = 0$. This requires that

$$\begin{aligned}\frac{\partial P[F]}{\partial p}(0, q) &= \frac{\partial G[F]}{\partial p}(0, q) + h(0, q)\frac{\partial w}{\partial p}(0, q) + w(0, q)\frac{\partial h}{\partial p}(0, q)\\ &= \frac{\partial F}{\partial p}(0, q)\end{aligned}$$

which implies that

$$h(0, q) = \frac{\dfrac{\partial F}{\partial p}(0, q) - \dfrac{\partial G[F]}{\partial p}(0, q)}{\dfrac{\partial w}{\partial p}(0, q)}.$$

After obtaining these values for all three edges, we define

$$h(x, y) = H[h](x, y)$$

where H is also an interpolation operator which only requires position values on ∂T. In other words, we are considering interpolants of the form

$$P[F] = G[F] + w \cdot H\left[\frac{F - G[F]}{w}\right]. \tag{3.30}$$

We now proceed to apply this technique on the on the triangular domain $T = T_s$ with $G = A^*$, as defined by equation (2.13), and $w(p, q) = pq(1 - p - q)$. While it is not necessary for the application of this technique, we assume for this example, C^2-compatibility on F. Performing the necessary calculations, we find that H will be operating upon a function with the boundary values

$$
\begin{aligned}
h(p,0) = \frac{1}{p(1-q)} \Big\{ & \frac{\partial F}{\partial q}(p,0) \\
& - p\Big[\frac{\partial F}{\partial q}(1,0) - \frac{\partial F}{\partial p}(1,0) + \frac{\partial F}{\partial p}(p,0) + 2F(1,0)\Big] \\
& - (1-p)\Big[\frac{\partial F}{\partial q}(0,0) + 2F(0,0)\Big] + 2F(p,0)\Big\} \\
h(p,1-p) = \frac{1}{p(p-1)} \Big\{ & \frac{\partial F}{\partial p}(p,1-p) \\
& - p\Big[\frac{\partial F}{\partial p}(1,0) - 2F(1,0)\Big] - 2F(p,1-p) - (1-p) \\
& \times \Big[\frac{\partial F}{\partial p}(p,1-p) - \frac{\partial F}{\partial q}(p,1-p) + \frac{\partial F}{\partial q}(0,1) - 2F(0,1)\Big]\Big\} \\
h(0,q) = \frac{1}{q(1-q)} \Big\{ & \frac{\partial F}{\partial p}(0,q) \\
& - q\Big[\frac{\partial F}{\partial p}(0,1) - \frac{\partial F}{\partial q}(0,1) + \frac{\partial F}{\partial q}(0,q) + 2F(0,1)\Big] \\
& - (1-q)\Big[\frac{\partial F}{\partial p}(0,0) + 2F(0,0)\Big] + 2F(0,q)\Big\}.
\end{aligned}
$$

We now note that this data is not C^0-compatible at the vertices. In fact,

$$
\begin{aligned}
\lim_{q\to 0} h(0,q) = & \frac{\partial^2 F}{\partial p\,\partial q}(0,0) - \frac{\partial F}{\partial p}(0,1) + \frac{\partial F}{\partial q}(0,1) + \frac{\partial F}{\partial q}(0,0) + \frac{\partial F}{\partial p}(0,0) \\
& + 2F(0,0) - 2F(0,1)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{p\to 0} h(p,0) = & \frac{\partial^2 F}{\partial p\,\partial q}(0,0) - \frac{\partial F}{\partial q}(1,0) + \frac{\partial F}{\partial p}(1,0) + \frac{\partial F}{\partial q}(0,0) + \frac{\partial F}{\partial p}(0,0) \\
& + 2F(0,0) - 2F(1,0).
\end{aligned}
$$

Similar conditions hold for the other two vertices. In order to continue this process, we use for H the following generalized version of A^* which can accomodate this type of incompatible data.

$$
\begin{aligned}
\bar{A}[F] = & \sum_{i=1}^{3} (1-b_i)^2 F(S_i) + \sum_{\substack{i=1 \\ i\neq j\neq k\neq i}}^{3} \frac{b_i^2}{1-b_i} \\
& \times \Big[b_j \lim_{\epsilon\to 0} F(V_i + \epsilon(V_j - V_i)) + b_k \lim_{\eta\to 0} F(V_i + \eta(V_k - V_i))\Big]
\end{aligned}
\tag{3.31}
$$

Performing all of the required calculations and mapping to the triangle T we obtain

$$\bar{P}[F] = \sum_{i=1}^{3} \bar{B}_i[F] + \sum_{\substack{i=1 \\ i\neq j\neq k\neq i}}^{3} \Big[F(V_i)[2[g(b_j, b_k) + g(b_k, b_j) + b_j b_k b_i^3] - b_i^2] + \frac{\partial F}{\partial e_j}(V_i)\, g(b_j, b_k) + \frac{\partial F}{\partial e_k}(V_i)\, g(b_k, b_j) - \frac{\partial^2 F}{\partial e_j\, \partial e_k}(V_i)\, b_j b_k b_i^3\Big] \tag{3.32}$$

where $\bar{B}_i$ is B_i of equation (3.7) based upon

$$h(t) = t^2(1 + 2t^2 - 3t^3), \qquad \bar{h}(t) = t^4(t-1)$$

and

$$g(s, t) = \frac{s^3 t^2(1 - s - t)}{1 - s} - t(1 - s - t)(1 - t)^3 - st(1 - s - t).$$

Since $\bar{P}$ is of the form given by equation (3.30), the precision of $\bar{P}$ will include the precision set of A^* along with $w = b_1 b_2 b_3$ and $w \cdot p$ for p in the precision set of $\bar{A}$. Thus, we have that $\bar{P}$ is exact for

$$\langle b_i, b_j b_k, b_i b_j b_k, b_i^2 b_j b_k, b_i b_j^2 b_k^2, i, j, k = 1, 2, 3;\ i \neq j \neq k \neq i \rangle.$$

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